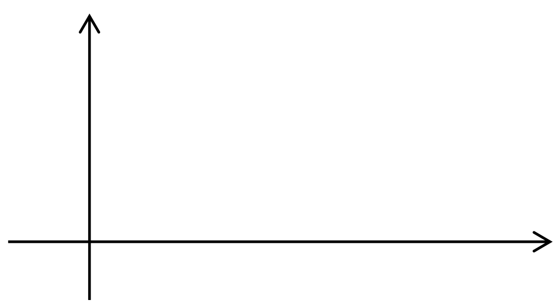
Area Under the Curve
Visual Aspect of "thinking backwards"

Ex 1) Suppose Madison is running at a speed of 8 miles per hour for 2 hours. Sketch the velocity/time graph in the space below. Make sure to label the axis with appropriate units.

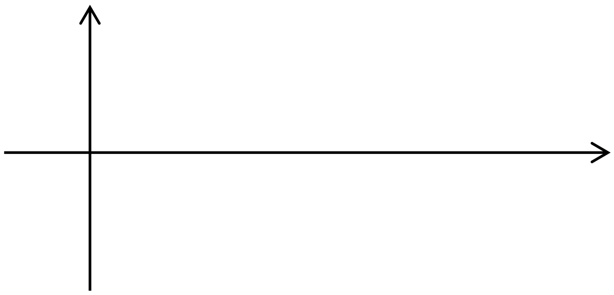
What is Madison's position after 2 hours from her starting point?

Ex 2) Suppose an object increases it speed 4 m/s every second. Sketch the velocity/time graph in the space below for the 1st 5 seconds. Make sure the label the axis with appropriate units.



What is the object position from its starting point after 5 seconds?

Ex 3) Suppose Ryan walked due north 5 feet per second in 2 seconds, then turned around and walked due south 5 feet per second in 2 seconds. Sketch the velocity/time graph in the space below for the 1st 4 seconds. Make sure to label the axis with appropriate units.



What is Ryan's position (displacement) from his starting point after 4 seconds?

Notice - Area:

Area Under a Curve Exploration

I. Estimating Area Under a Curve

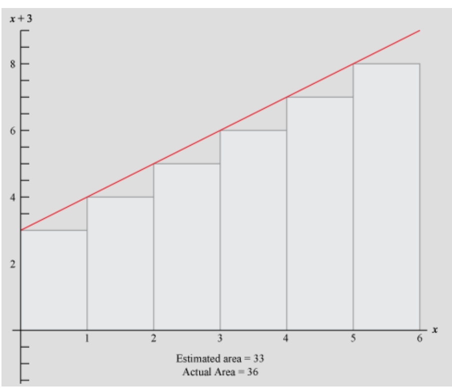
Two methods to estimate area under a curve on a closed interval **[a,b]** are Lower Sums and Upper Sums. Each consists of drawing equally spaced rectangles that lie above or below the curve. The process is straightforward if you think about two things:

1) The number of rectangles in the interval
2) The height of each rectangle.

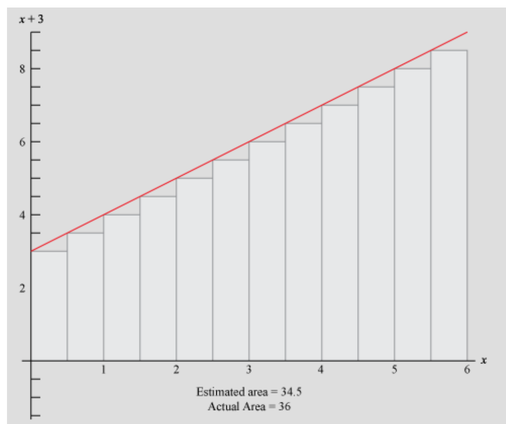
II. Number of Rectangles

In the example below, the function $f(x) = x + 3$ is restricted to the closed interval $[0,6]$. Normally you want the rectangles under the curve to be equally spaced. The first image has 6 equally spaced rectangles all having a width of 1. The second image has 12 equally spaced rectangles all having a width of $\frac{1}{2}$.

Rectangles Width 1

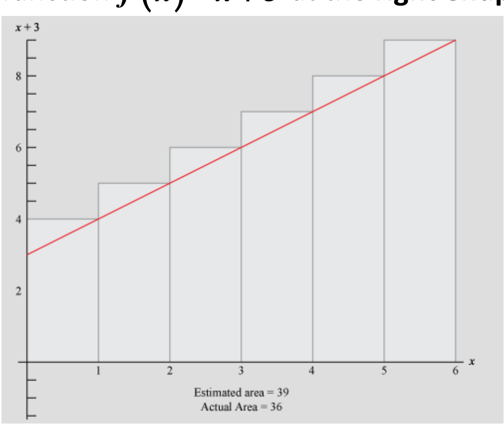


Rectangles Width $\frac{1}{2}$



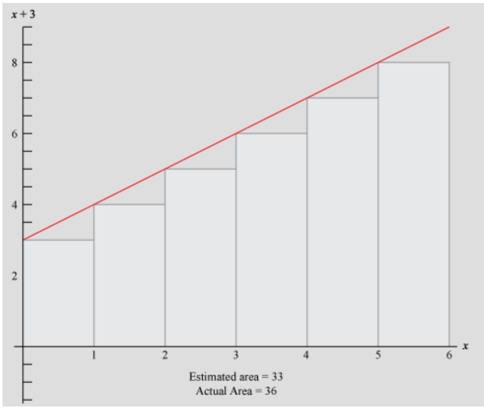
III. Height of Rectangles (Increasing Function)

To find the heights of the rectangles in an **Upper Sum**, the heights are determined by the function $f(x) = x + 3$ at the **right endpoint** of each rectangle. Look at the example below:



Height of 1st Rectangle:	$f(1) = 1 + 3 = 4$
Height of 2nd Rectangle:	$f(2) = 2 + 3 = 5$
Height of 3rd Rectangle:	$f(3) = 3 + 3 = 6$
Height of 4th Rectangle:	$f(4) = 4 + 3 = 7$
Height of 5th Rectangle:	$f(5) = 5 + 3 = 8$
Height of 6th Rectangle:	$f(6) = 6 + 3 = 9$

To find the heights of the rectangles in a **Lower Sum**, the heights are determined by the function $f(x) = x + 3$ at the **left endpoint** of each rectangle. Look at the example below:



Height of 1st Rectangle:	$f(0) = 0 + 3 = 3$
Height of 2nd Rectangle:	$f(1) = 1 + 3 = 4$
Height of 3rd Rectangle:	$f(2) = 2 + 3 = 5$
Height of 4th Rectangle:	$f(3) = 3 + 3 = 6$
Height of 5th Rectangle:	$f(4) = 4 + 3 = 7$
Height of 6th Rectangle:	$f(5) = 5 + 3 = 8$

IV. Estimating Area Using an Upper Sum and Lower Sum

Now we can use the information from III above to find an estimate of the area under the function $f(x) = x + 3$ on the interval from $[0,6]$ using Upper and Lower Sums.

Lower Sum Set Up–Using Left Rectangles

f(?)	Height of Rectangle	Width of Rectangle	Area of Rectangle
f(0)	3	1	3*1=3
f(1)	4	1	4*1=4
f(2)	5	1	5*1=5
f(3)	6	1	6*1=6
f(4)	7	1	7*1=7
f(5)	8	1	8*1=8

Lower Sum: 33
(Sum of all areas of rectangles in Lower Sum)

Upper Sum Set Up-Using Right Rectangles

f(?)	Height of Rectangle	Width of Rectangle	Area of Rectangle
f(1)	4	1	4*1=4
f(2)	5	1	5*1=5
f(3)	6	1	6*1=6
f(4)	7	1	7*1=7
f(5)	8	1	8*1=8
f(6)	9	1	9*1=9

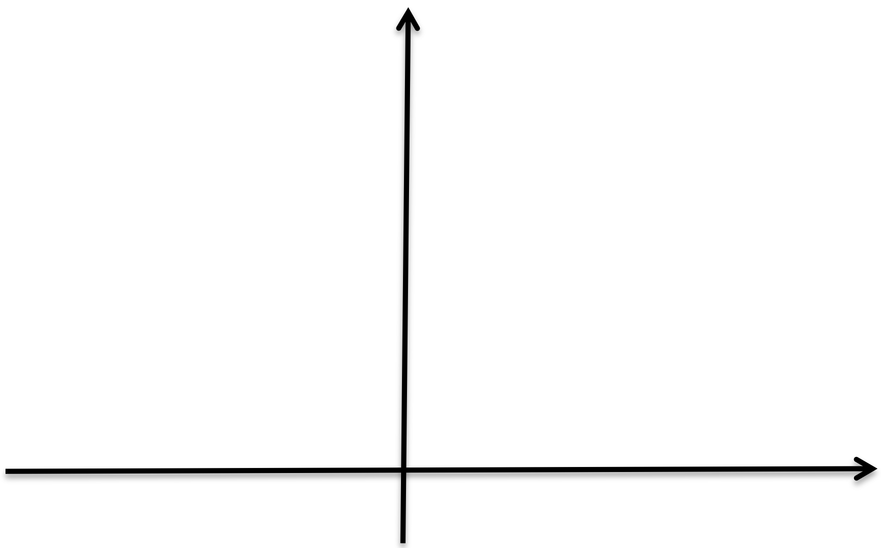
Upper Sum: 39
(Sum of all areas of rectangles in Upper Sum)

The actual area under $f(x) = x + 3$ on the interval from $[0,6]$ should lie somewhere between these two values! ***Find it by using geometry***.

$$\textit{Lower Sum} \le \textit{Area} \le \textit{Upper Sum}$$

You Try

1. Given y = -2x + 5 over [-2, 2], answer the following:
   a. Graph line bounded by the interval.



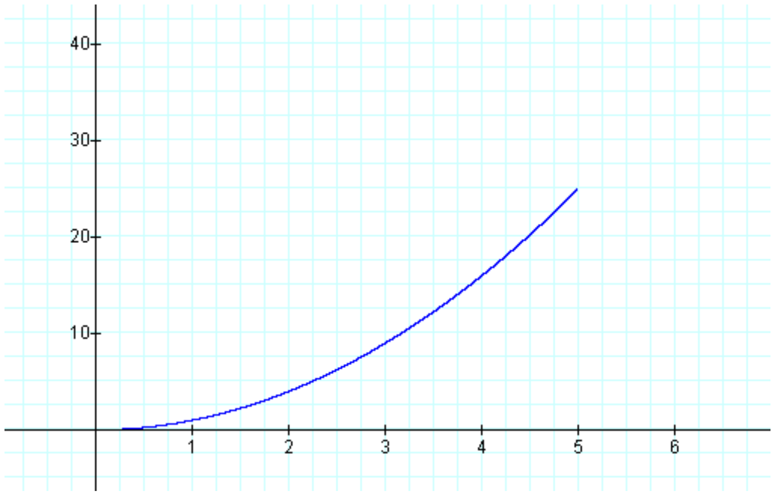
   b. Using geometry, find the area under the curve to the independent axis.

   c. Using 8 rectangles, what would be the base width of each rectangle?

   d. Use graph paper to draw the curve twice. Draw in 8 rectangles that represent the upper sum, then draw in 8 rectangles on the 2^{nd} curve that represent the lower sum. Find the upper and lower sum as done on the previous pages.

2. Approximate the area under the curve on the interval from [0,5]. Use lower and upper sums.

$f(x) = x^2$ on $[0,5]$



<u>Draw 5 equally spaced rectangles</u> on the interval from **[0,5].** For the lower sum, the rectangles should be below the curve (left endpoints). For the upper sum, the rectangles should be above the curve (right endpoints). Show the function evaluation for the heights.

<u>Lower Sum Set Up</u>

f(?)	Height of Rectangle	Width of Rectangle	Area of Rectangle
f(__)			
f(__)			
f(__)			
f(__)			
f(__)			

Lower Sum:____________________

<u>Upper Sum Set Up</u>

f(?)	Height of Rectangle	Width of Rectangle	Area of Rectangle
f(__)			
f(__)			
f(__)			
f(__)			
f(__)			

Upper Sum:____________________

If you had to guess at the actual area under the curve, what would it be?

Notes

How many rectangles would we use in order to find the actual area under any curve?

How would we do this (make sure to include calculus "lingo")?

The following topic will help us find area under non-linear curves.

Introduction to Summations

Summations will help us with calculating area under a curve. So, we are going to take a little side street and will come back to area under a curve.

Sequence: an ordered list of numbers whose domain is consecutive integers.

Ex) 1, 2, 3, 4, …..

Ex) 2, 4, 6, 8, 10….

Ex) 0, 1, 4, 9, 16….

Ex) 1, 3, 5, 7, 9….

Series: The sum of a sequence is a called a series.

Ex) 1 + 2 + 3 + 4 +…

Ex) 2 + 4 + 6 + 8 +…

***Domain**: the position of the term (1^{st}, 2^{nd}, 3^{rd}, and so on)

***Range**: the terms of the sequence (value we get back after plugging in 1, 2, 3, etc)

Finite Sequence: terminates, has a last term.

Infinite Sequence: continues without stopping.

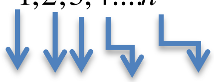
Example: Look at the following sequences

2, 4, 6, 8   2, 4, 6, 8,..

Finite Infinite

Notation: Sequences can be written using the following notation:

$1,2,3,4....n$



$a_1, a_2, a_3, a_4 .....a_n$

Example: If we have the following sequence

2, 4, 6, 8, 10, ……(etc)

Then
$$a_1 = 2$$
$$a_2 = 4$$
$$a_3 = 6$$
$$a_4 = 8$$
$$a_n = 2n$$

If a pattern can be found, you can represent the sequence as the following:

2, 4, 6, 8, 10, ……(etc) can be written as the following sequence:

$\{a_n\} = 2n$

Find the pattern and write as a sequence for the following:

1. 1, 2, 3, 4, 5,..

2. 1, 4, 9, 16, 25….

3. 0, 2, 4, 6, 8,..

4. 2, 5, 8, 11, 14…

5. 1, 3, 5, 7, 9, …

6. 1, 8, 27, 64, …

Summation Notation: (sigma notation). Greek letter sigma: Σ

Think about Microsoft Excel. There is a button that looks like sigma on the toolbar. If you use it, it will ADD cells together.

Finite Series: 2 + 4 + 6 + 8

End (this is the last value to input)

Summation notation: 2 + 4 + 6 + 8 = $\sum_{i=1}^{4} 2i$.

Start (input this number 1st)

Index of summation: *i* (n and k are also commonly used)

Lower limit: 1 (the first position)

Recall – domain is integers, so input 1, then 2, then 3, etc…

Upper limit: the last position (in the example above, it is 4)

$\sum_{i=1}^{4} 2i$ "The sum from *i* equal 1 to 4 of 2i"

Infinite Series: 2 + 4 + 6 + 8 + …

Summation notation for infinite series is very similar. If the terms go on FOREVER, then your upper limit changes to ∞. $\sum_{i=1}^{\infty} 2i$

Expand and simplify the following:

A. $\sum_{i=1}^{7} i^2$

B. $\sum_{n=0}^{5} \frac{2^n}{3n+1}$

C. $\sum_{n=3}^{6} (e^n - 4)$

Expand the 1st 4 terms and the last 4 terms of the following series.

$\sum_{i=1}^{50} i =$

Sum the 1st and last term (50th number)– what do you get?
Sum the 2nd and 49th term – what do you get?
Sum the 3rd and 48th term – what do you get?
Sum the 4th and 47th term – what do you get?
How many of these do you have?

Thus, what is the value of $\sum_{i=1}^{50} i$?

Let's try this again.
Expand the 1st 4 terms and the last 4 terms of the following series.

$\sum_{i=1}^{100} i =$

Sum the 1st and last term (100th number)– what do you get?
Sum the 2nd and 99th term – what do you get?
Sum the 3rd and 98th term – what do you get?
Sum the 4th and 97th term – what do you get?
How many of these do you have?

Thus, what is the value of $\sum_{i=1}^{100} i$?

So, find the pattern (find is the formula).

$$\sum_{i=1}^{n} i =$$

Does your formula work if i start at 0? Why or why not?

The following summation formulas may be helpful (which of these also work for i = 0?):

$$\sum_{i=a}^{n} c = c(n-a+1)$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

$$\sum \left(a_n \pm a_j\right) = \sum a_n \pm \sum a_j$$

<u>You Try:</u> Where you can use summation formulas, do it. Otherwise, expand out and simplify.

A. $\sum_{i=0}^{3}(4+\sqrt{9^i})$

B. $\sum_{i=0}^{150}(4i)$

C. $\sum_{i=0}^{150}(3)$

D. $\sum_{i=0}^{80}(5i-3i^2)$

E. $\sum_{i=0}^{300}(\frac{1}{i+1}-\frac{1}{i+2})$

F. $\sum_{i=0}^{550}(-4+2i-i^2)$

Summation and Area under Curve
Putting it all together

Ex1) Let's go back to page 101, $f(x) = x + 3$ over $[0,6]$.
Find the actual area using summations and limits.

Ex2) Now let's go back to page 103, y = -2x + 5 over [-2, 2].
Find actual area using summations and limits.

Ex3) From page 104, $f(x)=x^2$ on $[0,5]$. Find area using summations and limits.

Area Under Curve: Riemann Sums and Definite Integral

1. Use geometry to find the area under the curve of f(x) = x + 3 over [0,6].

2. A. Find $\int (x+3)\,dx$.

B. Plug in x = 0 and x = 6 into part 2A and find the difference of the two values. What do you notice between this answer and #1.

3. Using geometry, find the area under the curve of y = -2x + 5 over the interval [-2,2].

4A. Find $\int (-2x+5)\,dx$.

B. Plug in x = -2 and x = 2 into part A, find the difference between the 2 values. What do you notice between this and #3?

The Fundamental Theorem of Calculus:

Isn't that Cool!!!